Situation: Irrational Lengths

University of Georgia

Last Revised on August 1, 2013 by Sydney Roberts

Prompt

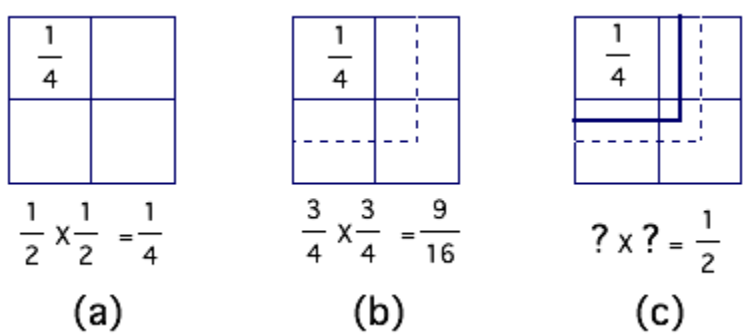
A secondary pre-service teacher was given the following task to do during an interview:

Given: square ABCD. Construct a square whose area is half the area of square ABCD. (Note: The pre-service teacher was not given a drawing or any dimensions for ABCD.)

The student chose the dimensions of ABCD to be 1 unit by 1 unit and approached the problem in two ways.

Method 1: Reasoning with a figure

She divided ABCD into smaller squares as shown in Figure 1a and noted each small square has area $\frac{1}{4}$. Sketching a new square as in Figure 1b, she claimed a $\frac{3}{4} \times \frac{3}{4}$ square has area $\frac{9}{16}$. She concluded that the square she wants (sketched in Figure 1c) has a side length somewhere between one-half and three-fourths.



Method 2: Reasoning from a formula

Assuming implicitly that the area of given square ABCD is 1 square unit, she noted that the desired area of the new square is one-half square unit. Using a formula for the area of a square, she produced $s^2 = \frac{1}{2}$

and then $s = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. After a long pause, she pointed to $\frac{\sqrt{2}}{2}$ and said, "I don't know how long that is, so I can't draw the square."

Commentary

Instead of focusing on the initial question that was posed in the prompt, this situation will revolve around the methods presented in the prompt. Mainly, it will be focused on irrational numbers that are derived from taking the square roots of positive integers. Therefore, the first two foci will discuss square roots of positive integers as irrational numbers and how to approximate their values, while the third focus will pertain to constructions of these lengths geometrically. These foci revolve around Method 2 from the prompt and in particular, the length of $\frac{\sqrt{2}}{2}$ that was derived by using the formula for area of a square. The last focus involves reasoning with a figure and how to construct a square such as the one asked for in the prompt.

Mathematical Foci

Focus 1

An irrational number is any number that can't be written in the form $\frac{a}{b}$ where a and b are integers and b is non-zero.

The Greeks thought of a "number" as being the measure of a "geometric length." Therefore, irrational numbers can be thought of as irrational lengths. Irrational lengths were first discovered by the Pythagoreans by investigating the diagonals of the unit square. A unit square has side lengths of 1, so following the Pythagorean theorem it was known that the diagonal d equaled $d = \sqrt{1^2 + 1^2} = \sqrt{1+1} = \sqrt{2}$. It was debated whether or not this length was rational and eventually Hippapus was able to prove that this length was in fact irrational. Refer to http://www.math.utah.edu/~pa/math/q1.html for a concise proof of this fact. Using similar logic, it can be shown that an irrational number multiplied or divided by a rational number results in an irrational number.

The square root of two is not the only irrational number, although it is a commonly known one. Other common irrational numbers are π, φ, and Euler's number, *e.* Because irrational numbers can't be written as a ratio of two integers, it is impossible to measure these lengths because no unit can fit evenly into these values. Most likely, this is why in Method 2, the pre-service teacher stated "I don't know how long that is" when referring to the length of $\frac{\sqrt{2}}{2}$.

However, this does not mean that the length cannot be constructed.

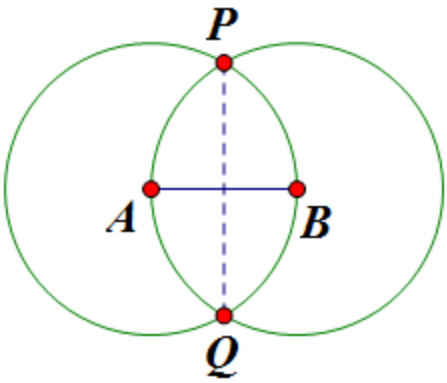
Focus 2

The Pythagoreans discovered that some irrational lengths can be constructed by using a compass and straightedge. More specifically, the square root of positive integers can be constructed, whether or not the value is rational or irrational.

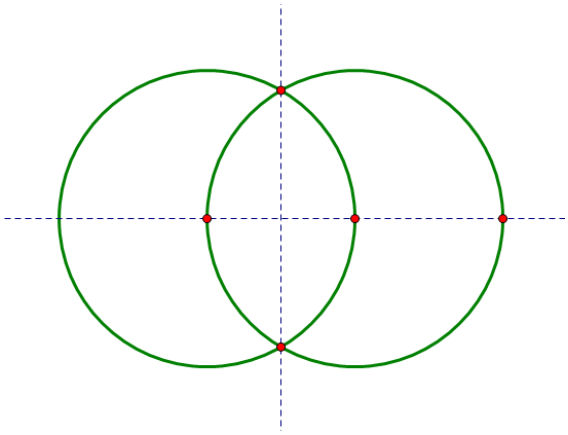
When discussing geometric constructions, it is important to be aware of Euclid's construction axioms. Euclid assumed that it is possible to:

1. Draw a straight line segment between any two points
2. Extend a straight line segment indefinitely
3. Draw a circle with given center and radius.

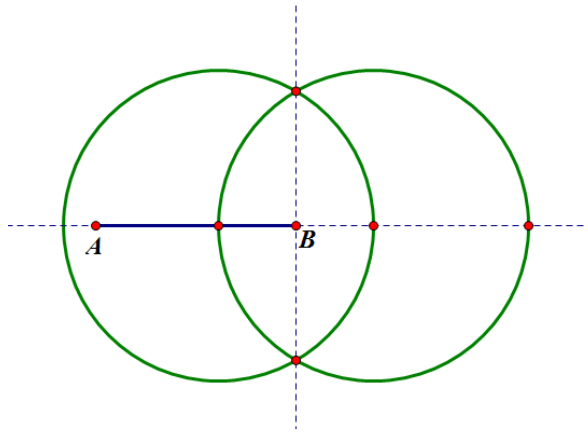
Using these axioms, one could construct perpendicular segments by finding the intersection points of two circles with equal radii as shown below where A and B are the centers of the two circles respectively. Then the line segment $\overline{PQ}$ would be perpendicular to $\overline{AB}$.



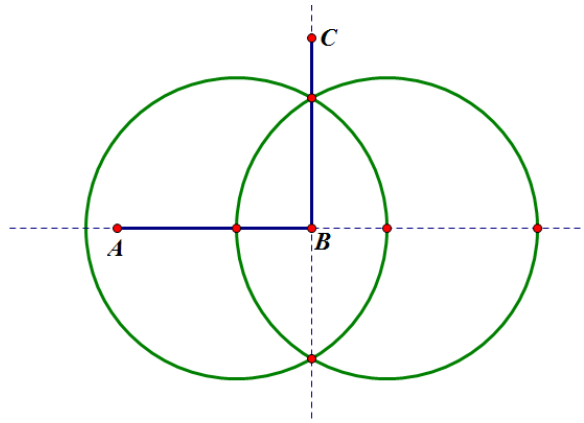
Using this, one could construct a square given a line segment with one unit length. Start by constructing two perpendicular lines, so that a 90° angle is present.



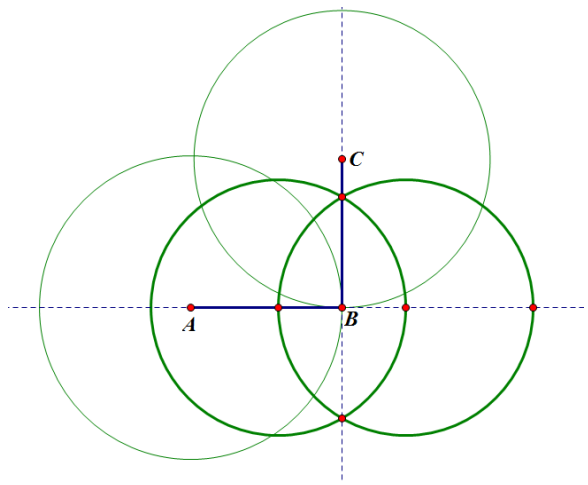
Then construct a line segment $\overline{AB}$ where one endpoint is at the constructed 90° angle. This segment will be the side length of the square, so one could set the compass to any radius and preserve this length (let's call it one unit).



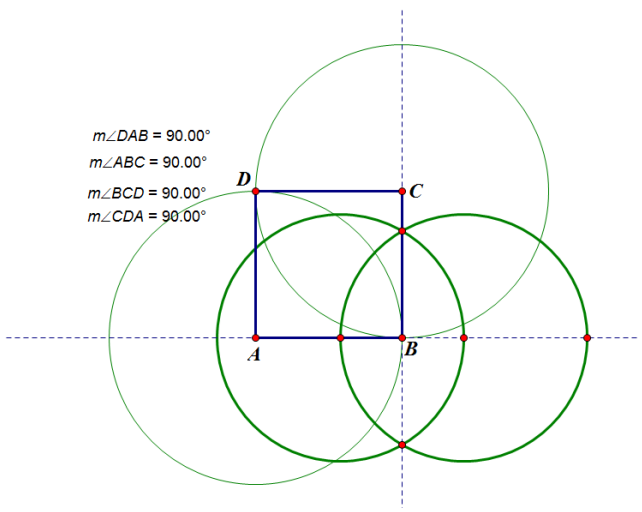
Now, preserving the length of segment $\overline{AB}$, copy that length along the perpendicular to $\overline{AB}$.



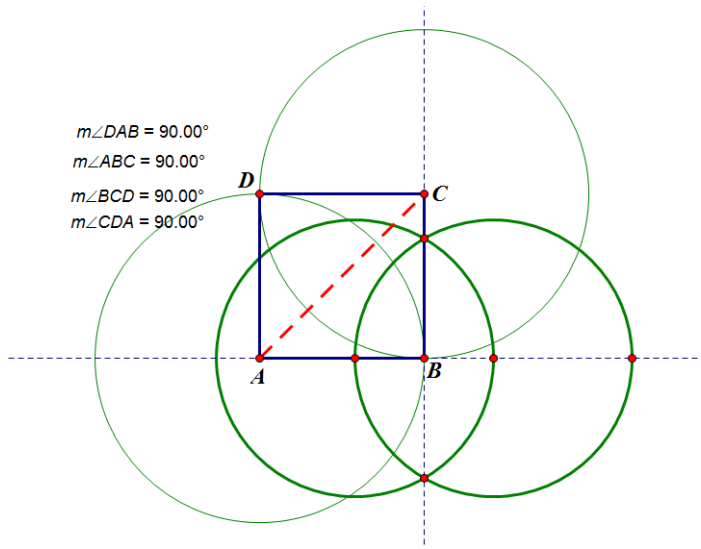
Now sense the length was preserved, $|AB| = |BC|$. Now keeping the compass set to the length of $\overline{AB}$. Construct to circles of radius $\overline{AB}$ around point A and point C. Where these two circles intersect will be the fourth corner of the square.



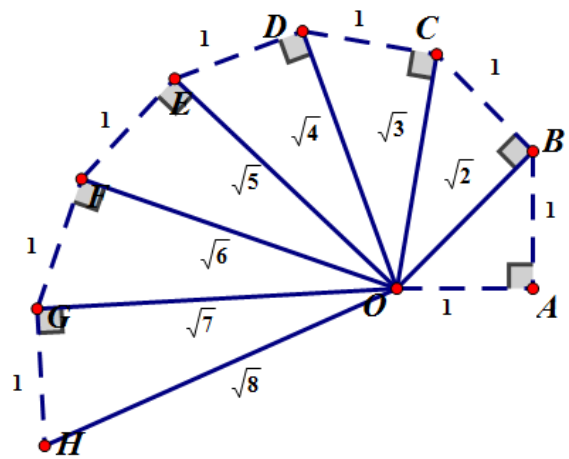
Now, construct a point at the intersection of the new circles, and construct straight line segments to points A and C. Then the square ABCD has been constructed.



Now using the straight edge, the diagonal of the square could be drawn, which is known to have a length of $\sqrt{2}$. The result of this shows that this irrational length can, in fact, be constructed even if it can't be measured.



Furthermore, there is a way to construct the square roots of integers by repeatedly constructing right triangles. Starting with a line segment of length $\sqrt{1} = 1$, then rotating this length by 90° to form a right triangle, an isosceles right triangle with base lengths of 1 and hypotenuse of $\sqrt{2}$ has been constructed. Then construct another leg of a right triangle with length of 1 attached to the $\sqrt{2}$ segment, the hypotenuse of this triangle would be $\sqrt{3}$. This construction can continue on and a spiral will start to form as the picture below shows.



Focus 3

The value of $\frac{\sqrt{2}}{2}$ *can be approximated using several methods. A simple hand calculator would present a good estimation, as well as using different algorithms and Newton's method.*

Using a calculator, one could punch in $\frac{\sqrt{2}}{2}$ and obtain a numerical approximation of 0.7071. However, this decimal doesn't terminate (a property of irrational numbers) and hence measuring this length would still prove to be rather difficult.

Another method to finding an estimate is using the "Divide-and-Average" method which is also known as the Babylonian Algorithm which dates back to 1700 B.C. This method involves starting with a guess of the square root of an integer, call it x, and then average the guess with the quotient of x and the guess. Take this new value and average it with the quotient of x and this value. This process can be repeated to continue getting a more accurate approximation. For example, consider finding an estimate for $\sqrt{2}$. Since the square root of 1 is 1, and the square root of 4 is 2, then the square root of two must lie somewhere between one and two. Let the guess be 1.5.

$$\sqrt{2} \approx 1.5$$

$$\sqrt{2} \approx \frac{1.5 + \frac{2}{1.5}}{2} = 1.41\bar{6}$$

$$\sqrt{2} \approx \frac{1.41\bar{6} + \frac{2}{1.41\bar{6}}}{2} = 1.4142 \ldots$$

Using a calculator, the square root of two is approximately 1.4142, so this method provides a very accurate estimate. One could then use the division algorithm to find $\frac{\sqrt{2}}{2} \approx \frac{1.4142}{2} = 0.7071$. For an explanation of why this algorithm works, refer to http://www.mathpath.org/Algor/squareroot/algor.babylon.htm.

Another method for finding square roots is the square-root algorithm which is much like the long division process. However, this process is long and tedious, and overall not as intuitive as the Babylonian Algorithm. For reference though, the following website provides a great outline of the process. http://www.homeschoolmath.net/teaching/sqr-algorithm-why-works.php

One could also use Newton's Method to approximate square root values. Essentially, Newton's Method finds approximates to the roots of a function. Newton's method is defined as a sequence that generates successively better approximations for the roots of the function. The algorithm for Newton's Method is as follows

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \; where \; n = 1,2,3,\cdots$$

$$\text{Let } x_0 \text{ be an initial guess}$$

To approximate $\sqrt{2}$, use the function that is presented in the prompt: $s^2 = {}^1/_2$. To define this as a function where the root would be $\sqrt{2}$, the function could be $f(x) = x^2 - \frac{1}{2}$. Then using $x_0 = 1.5$ as the initial guess like before, the algorithm yields

$$x_0 = 1.5$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{1.75}{3} = 0.91\bar{6}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.91\bar{6} - \frac{0.3402\bar{7}}{1.8\bar{3}} = 0.731\overline{06}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.731\overline{06} - \frac{0.03445}{1.46\overline{21}} \approx .707499$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} \approx .707499 - \frac{0.00055}{1.414998} \approx 0.70711$$

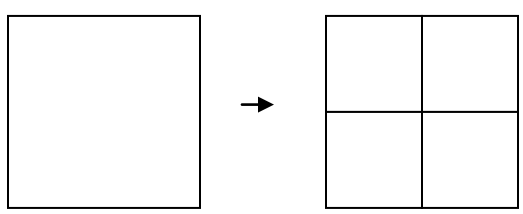
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} \approx .70711 - \frac{0.000004}{1.414218} \approx 0.70711$$

Repeating this algorithm will yield increasingly more accurate approximations, but at this point the successive approximations are remaining the same up to five decimal places, so it is safe to say that $\frac{\sqrt{2}}{2} \approx 0.70711$.

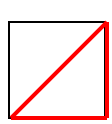
Focus 4

Given square ABCD that is divided into 4 smaller congruent squares, a diagonal of the smaller squares can be formed and has a length of $\frac{\sqrt{2}}{2}$.

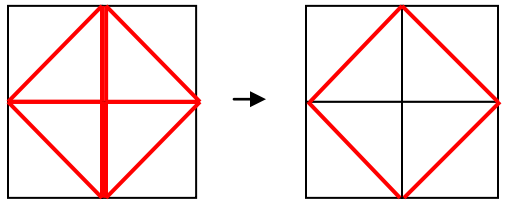
In Method 1, the pre-service teacher thought to take the 1x1 square ABCD and divide it into four smaller, congruent squares. As noted in the figure below. It was noted in the prompt that these smaller squares have an area of ¼ that of the original square.



The goal is to find a square that has the area of ½. Consider the triangle constructed by taking the diagonal of the square.



Note that by using the area of a triangle formula of $A = \frac{1}{2}bh = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$ that the area of this smaller triangle is ½ of the area of the smaller square. Now it would follow that by dividing each of the smaller squares in half by a triangle, we would have 4 figures that look as follows.



Is this new figure a square? Yes. Because the triangles were formed by cutting the smaller square in half and bisecting a corner of the square, the triangle is a 45°- 45°- 90° triangle with base lengths ½. Then the corner of the quadrilateral figure formed must be 45°+45°=90°. To find the side length of this figure, the Pythagorean Theorem can be used to see that the squared sums of the base legs are $(\frac{1}{2})^2 + (\frac{1}{2})^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ so the side lengths of the figure formed would be of equal length $s = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{\sqrt{2}}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. Then the area of this square is $\frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{2}{4} = \frac{1}{2}$ as desired with a side length of approximately 0.7071 which is in fact between ½ and ¾.

References

http://www.mathpath.org/Algor/squareroot/algor.square.root.htm

https://en.wikipedia.org/wiki/Irrational_number#Ancient_Greece

Stillwell, John. *The Four Pillars of Geometry*. New York: Springer, 2005. Print.